Quantization of a Complex Scalar field (Complex K-G fields):

A Real scalar field $\phi(x)$ satisfying the K-G eqn corresponds to electrically neutral particles.

Now, Complex scalar field where a conserved charge is associated with the field. for the real field this was not possible.

for complex K-G field, the Lagrangian density is replaced by

$$\mathcal{L} = N(\phi^{+}_{\alpha}\phi^{\alpha} - \mu^2\phi^{+}\phi) \quad --(1)$$

The field ϕ & its adjoint ϕ^{+} are treated as independent fields, leading to the K-G equations.

$$(\Box + \mu^2)\phi(x) = 0 \quad —— (2a)$$

$$\& \quad (\Box + \mu^2)\phi^{+}(x) = 0 \quad --(2b)$$

We know,

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{1}{c^2}\dot{\phi}^{+}(x) \quad --(3a)$$

$$\& \quad \pi^{+}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^{+}} = \frac{1}{c^2}\dot{\phi}(x) \quad —— (3b)$$

Equal time commutation relations become &

$$[\phi(x,t), \dot{\phi}^{+}(x',t)] = i\hbar c^2 \delta(x-x') \quad \text{—(4)}$$

$$[\phi(x,t), \phi(x',t)] = [\phi(x,t), \phi^{+}(x',t)] =$$
$$[\dot{\phi}(x,t), \dot{\phi}(x',t)] = [\dot{\phi}(x,t), \dot{\phi}^{+}(x',t)] =$$
$$[\phi(x,t), \dot{\phi}(x',t)] = 0 \quad \text{——(5)}$$

(∵ x & x′ are space-like with each-other)

||ly to the case of real scalar field, the fourier expansions of the fields are given by →

$$\phi(x) = \phi^{+}(x) + \phi^{-}(x)$$
$$= \sum_K \left[\frac{\hbar c^2}{2V\omega_K}\right]^{1/2} [a(k)e^{-ikx} + b^{+}(k)e^{ikx}] \quad \text{—(6)}$$

$$\boxed{\phi^{+} \rightarrow \text{adjoint of } \phi, \quad \phi^{+} \rightarrow \text{+ve freq. part of } \phi}$$

adjoint

$$\phi^{+}(x) = \phi^{++}(x) + \phi^{+-}(x)$$
$$= \sum_K \left[\frac{\hbar c^2}{2V\omega_K}\right]^{1/2} [b(k)e^{-ikx} + a^{+}(k)e^{ikx}] \quad \text{——(7)}$$

$\phi^{++}(x)$ & $\phi^{+-}(x)$ are the +ve & −ve frequency part of ϕ^{+}.

Using eqn (4), (5), (6) & (7), we obtain the commutation relations →

$$[a(k), a^{+}(k')] = [b(k), b^{+}(k')] = \delta_{kk'} \quad \text{——(8)}$$

& $[a(k), a(k')] = [b(k), b(k')]$

$= [a(k), b(k')] = [a^+(k), b(k)] = 0$

from these commutation relations → --(9)

$a(k)$, $b(k)$ → absorption or annihilation operators for two types of particles.

$a^+(k)$, $b^+(k)$ → creation operator for two types of particles.

Now, we shall define →

$$N_a(k) = a^+(k) a(k) \;\; \& \;\; N_b(k) = b^+(k) b(k) \quad \text{--(10)}$$

→ for a particle → for b particle

as the corresponding number operators with eigen values 0, 1, 2 - -

The vaccum state $|0\rangle$ is now defined by

$$a(k)|0\rangle = b(k)|0\rangle = 0 \;\; \forall \; k \quad \text{--(11)}$$

or equivalently, $\phi^+(x)|0\rangle = \phi^{\dagger +}(x)|0\rangle = 0 \;\forall\; x$ -

[∵ $\phi^+(x)$ depends upon

The energy momentum operator of the com

K-G field takes the form,

$$p^\alpha = \left(\frac{H}{c}, P\right) = \sum_k \hbar k^\alpha \left(N_a(k) + N_b(k)\right)$$

--(13)

Now, we turn to the charge.

The invariance of the Lagrangian density under the phase transformation follows the conservation of charge Q.

eq^n (14) & also su

$\mathcal{L}$ is invariant under the phase transformation implies

$$\phi_r \to \phi'_r = e^{i\varepsilon}\phi_r \approx (1+i\varepsilon)\phi_r = \phi_r + \delta\phi_r$$

$$\phi_r^+ \to \phi_r^{+\prime} = e^{-i\varepsilon}\phi_r^+ \approx (1-i\varepsilon)\phi_r^+ = \phi_r^+ + \delta\phi_r^+$$

where ε is a real parameter.

The charge Q has the form

$$Q = \frac{-iq}{\hbar c^2}\int d^3x\, N\left[\dot{\phi}^+(x)\phi(x) - \dot{\phi}(x)\phi^+(x)\right] \quad -(14)$$

The corresponding charge-current density is given by

$$s^\alpha(x) = \frac{-iq}{\hbar} N\left[\frac{\partial\phi^+}{\partial x_\alpha}\phi(x) - \frac{\partial\phi}{\partial x_\alpha}\phi^+(x)\right] \quad -(15)$$

This current density satisfies the continuity equation

$$\frac{\partial s^\alpha(x)}{\partial x^\alpha} = 0 \quad -(16) \qquad \left[J = \frac{I}{A} = \frac{dQ}{A\,dt}\right]$$

$\cancel{\frac{\partial Q}{A\,\partial t}}$

3

eqn (14) can be expressed in terms of creation & absorption operators as

$$Q = q \sum_{k} [N_a(k) - N_b(k)] \quad \text{---(17)}$$

such that $[H, Q] = 0$ i.e. Q commutes with H.

It is clear from eqn (17) that one must associate charge $+q$ & $-q$ with a & b particles, resp.

- Apart from the sign of charge a & b particles have identical properties.
- Interchange of a & b merely changes the sign of Q in eqn (17).
- This result is not restricted to spin 0 bosons but holds generally.
- This result indicate occurrence of antiparticle in association with all particles of non-zero charge.

Example:

Pair of charge π-mesons which include particle - antiparticle pair. Taking $q = e (>0)$

so, π^+ & π^- mesons.

- But in Real field charge operator Q is zero. corresponds to π^0.

Scanned with CamScanner

- The invariance of $\mathcal{L}$ under phase transformation would allow conservation of hypercharge. Dirac

* eg: K^0 particle & $\bar{K}_0$ antiparticle.

hypercharge $Y = \pm 1$. Rep. by complex K-G field.

Hypercharge is conserved in strong interaction.

Suppose ϕ_1 & ϕ_2 are real field.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$\phi^+ = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

Combination is complex field.

Dirac field :

Quantization formalism will be applied to the Dirac equation i.e to relativistic material particles of spin $1/2$ (fermions).

The commutation relation b/w absorption & creation operators for bosons will be replaced by anti-commutation relations for fermions.

Number Representation :

The representation in which states are specified by the occupation no. $n_r(K)$. It is of great practical importance in calculating transitions b/w initial & final states containing definite no. of particles with well defined properties.

Boson occupation no. can assume all values $n_r(K) = 0, 1, 2 --$

So, a modification will be required to describe particles obeying fermi-Dirac statistics (fermions) such as

electrons or muons for which ... occupation
no. is restricted to the values 0 and

No. Representation for fermions:

operators a_r & a_r^+, $r = 1, 2 \ldots$
satisfying the commutation relations:

$$[a_r, a_s^+] = \delta_{rs}$$

$$[a_r, a_s] = [a_r^+, a_s^+] = 0 \quad \text{--- (1)}$$

& operator $N_r = a_r^+ a_r$ --- (2)

operator Property:

$$[AB, C] = A[B, C] + [A, C]B \quad \text{--- (3)}$$

such that

$$[N_r, a_s] = [a_r^+ a_r, a_s] = a_r^+ [a_r, a_s] + [a_r^+, a_s] a_r$$

$$= 0 - \delta_{rs} a_r$$

$$= -\delta_{rs} a_r \quad \text{--- (4)}$$

& $$[N_r, a_s^+] = [a_r^+ a_r, a_s^+] = a_r^+ [a_r, a_s^+] + [a_r^+, a_s^+] a_r$$

$$= \delta_{rs} a_r^+ + 0$$

--- (5)

where

$a_r \rightarrow$ Absorption operator

$a_s^+ \rightarrow$ Creation operator

$N_r \rightarrow$ Number operator

& N_r has eigen value 0, 1, 2 ....

Vaccum state $|0\rangle$ is defined by

$$a_r|0\rangle = 0 \quad \forall \; r \qquad \text{--- (6)}$$

& the other states are built from the vaccum state as linear superposition of states of the form →

$$(a^+_{r_1})^{n_1} \, (a^+_{r_2})^{n_2} \, \text{---} \, |0\rangle \qquad \text{--- (7)}$$

Anticommutator bracket are given as →

$$\{A, B\} \text{ or } [A, B]_+ = AB + BA \qquad \text{--- (8)}$$

also $[AB, C] = A[B, C]_+ - [A, C]_+ B$ --- (9)

$$[a_r, a_s^+]_+ = \delta_{rs} \quad , \quad [a_r, a_s]_+ = [a_r^+, a_s^+] = 0 \qquad \text{--- (10)}$$

also, $(a_r)^2 = (a_s^+)^2 = 0$ --- (11)

We now have →

$$N_s^2 = a_s^+ a_s a_s^+ a_s$$
$$= a_s^+ (1 - a_s^+ a_s) a_s$$
$$= a_s^+ a_s - \underbrace{a_s^+ a_s^+ a_s^2}_{0}$$
$$= N_s$$
$$N_s^2 - N_s = 0$$
$$N_s(1 - N_s) = 0$$
$$\Rightarrow N_s = 0 \ \& \ N_s = 1$$

$\Rightarrow$ No. operator N_s has eigen value 0 & 1.

The state in which one particle is in the state s is defined by $|1_s\rangle = a_s^+ |0\rangle$ — (12)
→ vaccum state

for the two particle states, we have from the commutation relations (for $s \neq s$)

$$|1_s 1_s\rangle = a_s^+ a_s^+ |0\rangle = -a_s^+ a_s^+ |0\rangle = -|1_s 1_s\rangle \quad \text{--(13)}$$

$$[\because [a_s^+, a_s^+]_+ = a_s^+ a_s^+ + a_s^+ a_s^+ = 0]$$
$$\Rightarrow a_s^+ a_s^+ = -a_s^+ a_s^+$$

$\Rightarrow$ that the state $|1_s 1_s\rangle$ is antisymmetric under interchange of particles labels as required by fermions.

for $s = s$, $|1_s 1_s\rangle = (a_s^+)^2 |0\rangle = 0$

two particles can't exist in the same single particle state.

Covariant Perturbation Theory:

In this theory, we start with an explicity covariant formulation of classical electrodynamics in which all four component of the four vector $A^{\mu}(x) = (\phi, A)$ are treated on equal footing. (treated independently)

- This corresponds to introducing more dynamical degree of freedom than the system posses & these later removed by imposing suitable constraints.
- All four components of the four field vector $A^{\mu}(x)$ are than quantized.

1> Classical fields:

To express Maxwell's eqn (Partial differential eqn) in covariant form, we introduce the antisymmetric field tensor.

$$F^{\mu\nu} = \begin{array}{c|cccc|c} \nu & 0 & 1 & 2 & 3 & \mu \\ \hline & 0 & E_x & E_y & E_z & 0 \\ & -E_x & 0 & B_z & -B_y & 1 \\ & -E_y & -B_z & 0 & B_x & 2 \\ & -E_z & B_y & -B_x & 0 & 3 \end{array} \quad --(1)$$

Maxwell eqn

$$\left\{ \begin{array}{l} \nabla \cdot D = \rho \\ \nabla \cdot B = 0 \\ \nabla \times E = -\dfrac{\partial B}{\partial t} \\ \nabla \times H = \dfrac{\partial D}{\partial t} + J \end{array} \right.$$

Maxwell eqn in term of $F^{\mu\nu}$ & charge current density →

$$\partial_\nu F^{\mu\nu}(x) = \frac{1}{c} S^\mu(x) \quad --(2)$$

$$\partial^\lambda F^{\mu\nu}(x) + \partial^\mu F^{\lambda\nu}(x) + \partial^\nu F^{\lambda\mu}(x) = 0 \quad --(3)$$

The field $F^{\mu\nu}$ can be expressed in term of four vector potential

$$A^\mu(x) = (\phi, A) \text{ by}$$

$$F^{\mu\nu}(x) = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x) \quad --(4)$$

Using eqn (4) in eqn (2), we have

$$\square A^\mu(x) - \partial^\mu(\partial_\nu A^\nu(x)) = \frac{1}{c} S^\mu(x) \quad -(5)$$

$\boxed{\square = \partial_\nu \partial^\nu}$

These eqn are lorentz covariant & they are also invariant under guage transform

$$A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu f(x) \quad -$$

The eqn (5) can be derived from the lagrangian density. (which is suitable for quantization)

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu A_\mu(x))(\partial^\nu A^\mu(x)) - \frac{1}{c} S_\mu(x) A^\mu \quad --(7)$$

Conjugate field (from 7)

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\frac{1}{c^2}\dot{A}^\mu(x) \quad --(8)$$

which all are non-vanishing, so the canonical quantization formalism can be applied. (2)

field eqn from eqn (7)

$$\Box A^{\mu}(x) = \frac{1}{c} s^{\mu}(x) \quad \text{---} (9)$$

{eqn (9) equivalent to Maxwell eqn if the potential $A^{\mu}(x)$ satisfies the constraint}

$$\partial_{\mu} A^{\mu}(x) = 0 \quad \text{--} (10)$$

(In free field case ($s^{\mu}(x) = 0$), so eqn (9) become) →

$$\Box A^{\mu}(x) = 0 \quad \text{--} (11)$$

→ eqn (11) enables us to expand the free E.M field $A^{\mu}(x)$ in a complete set of soln of the wave eqn.

$$A^{\mu}(x) = A^{\mu +}(x) + A^{\mu -}(x) \quad \text{---} (12)$$

where,

$$A^{\mu +}(x) = \sum_{rk} \sqrt{\frac{\hbar c^2}{2VW_k}}\, \varepsilon_{r}^{\mu}(k)\, a_r(k)\, e^{-ikx} \quad \text{---} (13)$$

$$A^{\mu -}(x) = \sum_{rk} \sqrt{\frac{\hbar c^2}{2VW_k}}\, \varepsilon_{r}^{\mu}(k)\, a_r^{+}(k)\, e^{ikx} \quad \text{---} (14)$$

Summation of (13) + (14) wave vector k allowed by periodic boundary condition.

$$k^0 = \frac{1}{c} W_k = |K| \quad \text{---} (15)$$

Summation over ($r=0$ to $r=3$) correspo[nds] to the fact that for the four-vector field $A^{\mu}(x)$ there exists, for each k four linearly independent polarization states.

* Commutation Rules for Covariant Quantization;

The free E.M field is quantized, using the Lagrangian densit[y] given by eqⁿ (7) with $S_{\mu}(x) = 0$.

Equal time commutation relation.

$$[A^{\mu}(x,t), A^{\nu}(x',t)] = 0$$

$$[\dot{A}^{\mu}(x,t), \dot{A}^{\nu}(x',t)] = 0$$

$$[A^{\mu}(x,t), \dot{A}^{\nu}(x',t)] = -i\hbar c^{2} g^{\mu\nu} \delta(x-x') \quad \text{--- (16)}$$

Covariant commutation relations for the $A^{\mu}(x)$ are given as -

$$[A^{\mu}(x), A^{\nu}(x')] = i\hbar c D^{\mu\nu}(x-x') \quad \text{--- (17)}$$

$$D(x) = \lim_{\mu \to 0} \left[-g^{\mu\nu} \Delta(x) \right]$$

here, $\Delta(x) \rightarrow$ invariant Δ-function.

To gain the photon interpretation of the quantized field, we get

$$[a_r(k), a_s^{+}(k')] = l_r \delta_{rs} \delta_{kk'}$$

$$[a_r(k), a_s(k')] = [a_r^{+}(k), a_s^{+}(k')] = 0 \quad \text{--(18)}$$

from eq^n (15) $l_r = 1$ for $r = 1, 2, 3$.

$r = 1, 2 \rightarrow$ for transverse photon
$r = 3$ longitudinal photon
$r = 0$ scalar photon

* Quantization of E.M. field:

(Gupta - Bleuler formalism)

In this formalism, the operators $a_r(k)$ ($r = 1, 2, 3$) & 0 are interpreted as absorption operator & creation operation

$$a_r^{+}(k), r = 1, 2, 3 \text{ \& } 0$$

The vaccum state $|0\rangle$ is that state in which there is no photon of any kind present.

i.e $a_r(k)|0\rangle = 0 \;\forall\; k,\; r = 0, 1, 2, 3$

or $A^{\mu+}(x)|0\rangle = 0 \;\forall\; x,\; \mu = 0 \text{ -- } 3$

The operator $a_r^{+}(k)$ operating on the vaccum state $|0\rangle$ create one photon state.

$$a_r^{+}(k)|0\rangle = |1_{kr}\rangle$$

in which one transverse $(r = 1, 2)$, or longitudinal $(r = 3)$ or scalar $(r = 0)$, photon of momentum $\hbar k$ is present.

* The Hamiltonian operator of the field is given by →

$$H = \int d^3x \, N\left[\pi^{\mu}(x) \dot{A}_{\mu}(x) - \mathcal{L}(x)\right]$$

On substituting the free lagrangian density corresponding to eqn (7) & eqn (8)

state $|$ $\eta^{\mu}(x)$ & eqn (12) for the expansion (4)
for the field we have :

$$H = \sum_{\lambda k} \hbar \omega_k \, l_\lambda \, a_\lambda^{+}(k) a_\lambda(k)$$

$$H|1_{k\lambda}\rangle = \sum_{q s} \hbar \omega_q \, l_s \, a_s^{+}(q) \, a_s(q) \, a_\lambda^{+}(k) |0\rangle$$

$$= \hbar \omega_k \, a_\lambda^{+}(k) |0\rangle$$

$\lambda = 0 \ldots 3$

i.e the energy has true value $\hbar \omega_k$ for transverse, longitudinal & scalar photons.

Corresponding the no. operator is

$$N_\lambda(k) = l_\lambda \, a_\lambda^{+}(k) a_\lambda(k)$$

&
$$\langle \psi | H | \psi \rangle = \langle \psi | \sum_{k} \sum_{\lambda=1}^{2} \hbar \omega_k \, a_\lambda^{+}(k) a_\lambda(k) | \psi \rangle$$

transverse $\lambda = 1, 2$

$\Rightarrow$ only the transverse photon contribute to the expectation value of energy.

$\Rightarrow$ The longitudinal & scalar photons are not absorbed as free particle.

- for free field (i.e no charges present the vaccum is represented by the state $|0\rangle$ in which no photons of any kind are present.

But the vaccum could be described by any state containing no transverse & only allowed admitance of scalar & longitudinal photons.

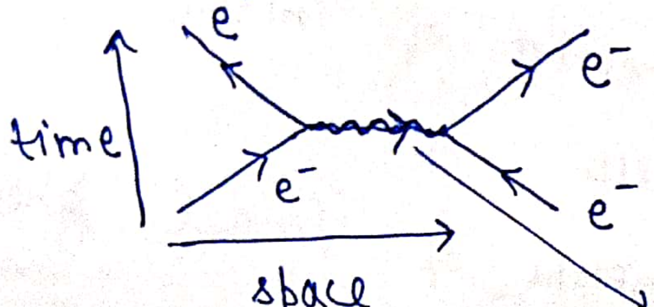
- for the EM field in the presence of charges, we can no longer ignore the logitudinal & scalar photons.

Feynman Diagram:

Every feynman diagram are pictorial representa of mathematical expression describing the behaviour of subatomic particles.

Case-I →

Electron - Electron scattering:



→ Represent the energychange or exchange particle which has a brief or virtual existence. (In this case photon)

Fermion

Photon

Gluon

Boson (W, Z)

Let P_1, P_2 & P_1', P_2' be four momenta of the initial & final e^- respectively. r_1, r_2 & r_1', r_2' their spin projections.

The intial & final state vectors are:

$$|P_1, r_1; P_2, r_2\rangle = C^+_{r_1}(P_1)\, C^+_{r_2}(P_2)\,|0\rangle$$

$$|P_1', r_1', P_2', r_2'\rangle = C^+_{r_1'}(P_1')\, C^+_{r_2'}(P_2')\,|0\rangle$$

Example → 2

Scattering of a photon by an electron (Compton scattering):

Consider a process in which a photon is scattered by an e^-.

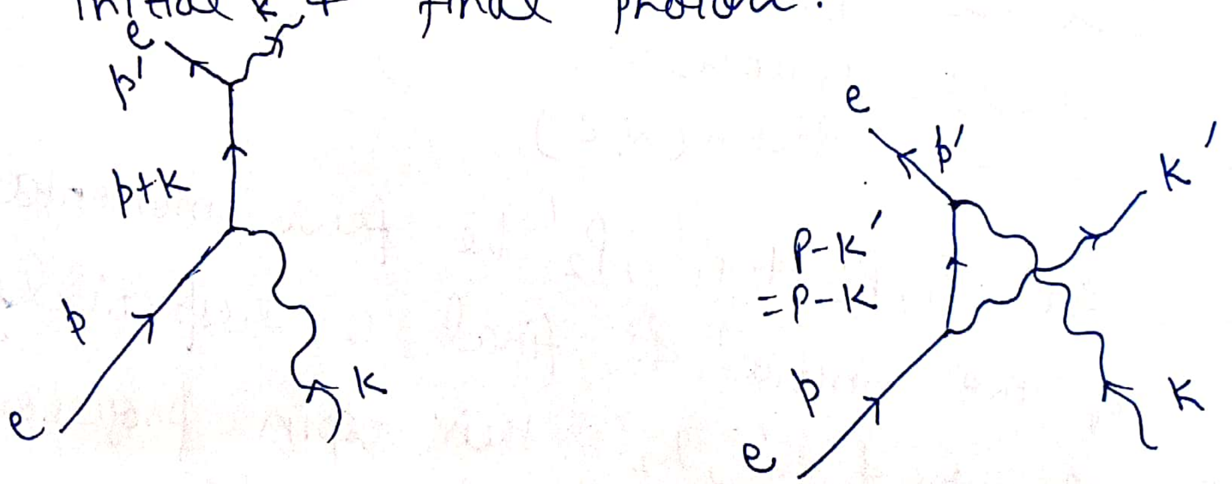
$$\gamma + e \rightarrow \gamma + e$$

Let k & P Momenta of initial photon & e^- resp.

& k' & p' Momenta of final photon & e^- resp.

& r & r' spin projections of the initial & final electron.

& λ & λ' polarization vectors of initial & final photon.



Compton scattering

... & final state vectors are:

$$|P, r, K, \lambda\rangle = C_r^{+}(P)\, a_\lambda^{+}(k)\, |0\rangle$$

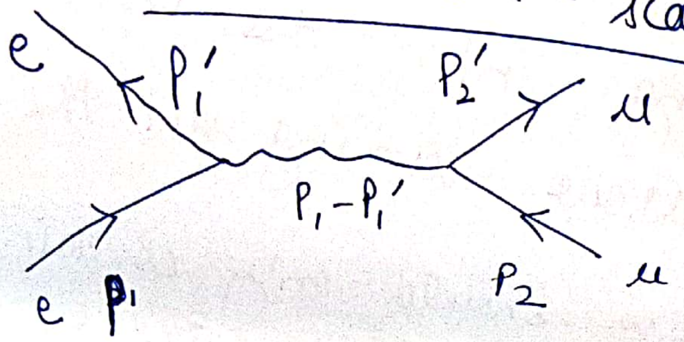
$$|P', r', k', \lambda'\rangle = C_{r'}^{+}(P')\, a_{\lambda'}^{+}(k')\, |0\rangle$$

Matrix element are given as →

$$\langle P', r', k', \lambda' | S | P, r, k, \lambda \rangle$$

example - 3

Electron muon - scattering:



Let P_1 & P_2 initial momenta of proton & muon resp.
& P_1' & P_2' final momenta of proton & muon resp.

Rules for writing Feynman Diagram:

1) You can draw two kinds of lines, a straight line with an arrow or a wiggly line. → ~~~